1st lecture 1601.03586 with Braverman, 2nd 1503.03676
Finkelberg 3rd: examples

G_c : compact Lie group
G : its complexification
M : quaternionic representation of G_c
= symplectic repr. of G

$$(G, M) \underset{\text{physics}}{\rightsquigarrow} \left(\boxed{\text{4d } N=2 \text{ SUSY gauge theory}} \right)$$
$$\downarrow /S^1$$
$$\boxed{\text{3d } N=4 \text{ SUSY gauge theory}} \rightsquigarrow \text{Coulomb / Higgs branches}$$

↑
hyperkähler mfd
possibly with singularities
with $SU(2)$-action rotating cpx str's

<u>Problem</u> Construct those branches in a mathematically rigorous way.

☆ Higgs branch

<u>Ans</u>. $M /\!/\!/ G_c$ hyperkähler quotient
$= \mu_{\mathbb{C}}^{-1}(0) /\!/ G$ $\quad \mu = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}}) : M \to \operatorname{Lie} G_c \otimes \operatorname{Im} \mathbb{H}$
hK moment map

$(G_c \curvearrowright M \curvearrowleft \mathbb{H} \circlearrowleft_{Sp(1)=SU(2)})$

☆ Coulomb branch $\mathcal{M}_C = \mathcal{M}_C(G, M) \leftarrow$ much more difficult!
expected properties
- $\dim \mathcal{M}_C = 2 \operatorname{rank} G$

$\mathcal{M}_C \underset{\text{birational}}{\approx} T^*T^\vee / W$ $\quad T \subset G$ max torus

$T^\vee =$ dual torus

classical description $\quad W$: Weyl group

- $\mathcal{M}_C \curvearrowleft SU(2)$ rotating cpx structures $I . J . K$

$\quad \overset{\cup}{S^1} \quad$ fixing I

<u>NB.</u> $\mathcal{M}_C$ is <u>not</u> a cone in general

- (physical "definition")

Suppose $\mathcal{M}_H = \{0\}$ (e.g. $M = 0$)

then $\mathcal{M}_C$ is smooth and

underlying Rozansky-Witten theory

gauge theory $(G, M) \cong$ 3d SUSY σ-model with target $\mathcal{M}_C$

If $\mathcal{M}_H \neq \{0\}$, $\mathcal{M}_C$ has singularities.

Then RHS must be defined carefully, ...

(e.g. taking resolution by matters
deformation

Important examples

⓪ $M = 0$ $\quad \mathcal{M}_C = T^*G^\vee /\!/\!/_{(\chi,\chi)} N^\vee \times N^\vee$

① $M = \mathfrak{g} \oplus \mathfrak{g}^*$ (adjoint $\oplus$ coadjoint) $\quad (\mathcal{M}_C = T^*T^\vee / W)$

② $G = T$ torus abelian

s.t. $1 \to T \to (\mathbb{C}^\times)^n \to T_F \to 1$ $\quad M = \mathbb{C}^n \oplus (\mathbb{C}^n)^*$

$\mathcal{M}_H = M /\!/\!/ T_c =$ toric hK mfd

$\mathcal{M}_C =$ dual hK mfd $= M /\!/\!/ T_F^\vee$

$1 \to T_F^\vee \to (\mathbb{C}^\times)^n \to T^\vee \to 1$

③ quiver gauge theory

$Q=(Q_0,Q_1)$: quiver

$V=\bigoplus_{i\in Q_0} V_i$, $W=\bigoplus_{i\in Q_0} W_i$

$$N=\bigoplus_{h\in Q_1}\mathrm{Hom}(V_{o(h)},V_{i(h)})\oplus\bigoplus_{i\in Q_0}\mathrm{Hom}(W_i,V_i)$$

$M=N\oplus N^*$, $G=\prod_{i\in Q_0}GL(V_i)$

($\mathcal{M}_H$ = quiver variety) $\mathcal{M}_C$ = 3rd lecture

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Mathematically rigorous definition when $M=N\oplus N^*$
as an affine algebraic variety with Braverman Finkelberg

1°. $(G,N)\rightsquigarrow \mathcal{R}\equiv\mathcal{R}_{G,N}\curvearrowleft G_\mathcal{O}=G[[z]]$
(∞-dimensional) moduli space

2° convolution product on $H_*^{G_\mathcal{O}}(\mathcal{R})$

3° $\mathcal{M}_C\overset{\text{def.}}{=}\mathrm{Spec}\,H_*^{G_\mathcal{O}}(\mathcal{R})$ study properties e.g. commutative

1°. $G_K=G((z))\supset G_\mathcal{O}=G[[z]]$ $D=\mathrm{Spec}\,\mathbb{C}[[z]]\supset D^\times=\mathrm{Spec}\,\mathbb{C}((z))$

$\mathrm{Gr}_G=G_K/G_\mathcal{O}$: affine grassmannian

$=\{(\mathcal{P},\varphi)\mid\mathcal{P}: G\text{-b'dle over } D,\ \underset{\text{trivialization}}{\varphi:\mathcal{P}|_{D^\times}\simeq \overset{\mathcal{P}_{\text{triv}}}{\overset{\|}{D^\times\times G}}}\}/\text{isom}$

λ : coweight of G $\mathrm{Hom}(\mathbb{C}^\times,T)$, $T\subset G$ $z^\lambda\in\mathrm{Gr}_G$

Fact $\mathrm{Gr}_G=\bigsqcup_{\lambda:\text{ dominant coweight}}\underbrace{G_\mathcal{O}\cdot z^\lambda}_{=:\,\mathrm{Gr}_G^\lambda}$ $\overline{\mathrm{Gr}_G^\lambda}$: f.d. proj. scheme

$\overline{\mathrm{Gr}_G^\lambda}=\bigsqcup_{\mu\leq\lambda}\mathrm{Gr}_G^\mu$

$N_{\mathcal{O}} = N[[z]]$

$\mathcal{T} = G_{\mathcal{K}} \times_{G_{\mathcal{O}}} N_{\mathcal{O}} = \{ (\mathcal{P}, \varphi, s) \mid (\mathcal{P}, \varphi) : \text{as above},\ s \in H^0(\mathcal{P} \times_G N) \}$

vector b'dle over Gr_G, fiber $= N_{\mathcal{O}}$

$\mathcal{R} = \{ (\mathcal{P}, \varphi, s) \in \mathcal{T} \mid \varphi(s) \in N_{\mathcal{O}} \}$

$= \{ \mathcal{P}_{\mathrm{triv}}, \mathcal{P}, \varphi : \mathcal{P}|_{D^\times} \xrightarrow{\sim} \mathcal{P}_{\mathrm{triv}}|_{D^\times},\ s \in H^0(\mathcal{P} \times_G N)$ s.t. $\varphi(s) \in H^0(\mathcal{P}_{\mathrm{triv}} \times_G N) \} /_{\mathrm{isom}}$

$\mathcal{R} \curvearrowleft G_{\mathcal{O}} \ni h$ $\quad h \cdot [g, s] = [hg, s]$ change the trivialization

$\mathcal{R} \to \mathrm{Gr}_G$ not a vector b'dle, but it is so over Gr_G^λ

$\cap$ $\mathcal{T}$

$\mathcal{R}_\lambda := \mathcal{R}|_{\mathrm{Gr}_G^\lambda} \subset \mathcal{T}|_{\mathrm{Gr}_G^\lambda}$ finite corank

2° $H_*^{G_{\mathcal{O}}}(\mathcal{R})$

$\nwarrow$ relative to $\mathcal{T}$ e.g. $\deg \mathcal{R}_{[g]} = -2 \operatorname{codim} \mathcal{R}_{[g]} \text{ in } \mathcal{T}_{[g]}$

Rem well-defined (stabilize)

Prop $\exists$ explicit combinatorial formula (monopole formula) of Poincaré (polynomial) of $H_*^{G_{\mathcal{O}}}(\mathcal{R})$

$\because$ $\mathcal{R} = \bigcup \mathcal{R}_{\leq \lambda}$ $\qquad \mathcal{R}_\lambda \xrightarrow[\text{vect b'dle}]{} \mathrm{Gr}_G^\lambda$ $\nwarrow$ vector b'dle over generalised flag var.

- convolution product

heuristic definition $\begin{array}{ccc} \mathcal{T} & \longrightarrow & N_K \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [g,s] & \longmapsto & gs \end{array}$ or $\varphi(s)$

$$\mathcal{T}\times_{N_K}\mathcal{T} = \{(\mathcal{P}_1,\varphi_1,s_1,\mathcal{P}_2,\varphi_2,s_2) \mid \varphi_1(s_1)=\varphi_2(s_2)\}$$

$\mathcal{R} = \{\mathcal{P}_2 = \mathcal{P}_{\mathrm{triv}}, \varphi_2 = \mathrm{id}\}$ $\Big($ $\mathcal{T}\times_{N_K}\mathcal{T}/G_\mathcal{O} = \mathcal{R}$

$$\begin{tikzcd}
 & \mathcal{T}\times_{N_K}\mathcal{T}\times_{N_K}\mathcal{T} \arrow[dl, "p_{12}"'] \arrow[d, "p_{23}"'] \arrow[dr, "p_{13}"] & \\
\mathcal{T}\times_{N_K}\mathcal{T} & \mathcal{T}\times_{N_K}\mathcal{T} & \mathcal{T}\times_{N_K}\mathcal{T}
\end{tikzcd}$$

$$\alpha*\beta \overset{\text{def.}}{=} p_{13*}(p_{12}^*(\alpha)\cap p_{23}^*(\beta))$$

$\Big($ **Prob** $\mathcal{T}$: not smooth
$\mathcal{T}\times_{N_K}\mathcal{T}$: too ∞-dimensional

$*$ preserves the homological grading $\therefore$ graded algebra

3°. **Prop.** $*$ is commutative

1° Use BD grassmannian
2° reduction to abelian case $\longrightarrow$ explained later

Def. $\mathcal{M}_C = \mathrm{Spec}(H_*^{G_\mathcal{O}}(\mathcal{R}), \text{conv. product})$

Ex1 $G = \mathbb{C}^\times$, $N = 0$

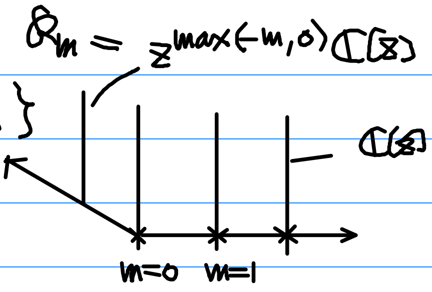
$$\mathcal{R} = \mathrm{Gr}_{\mathbb{C}^\times} \simeq \mathrm{Hom}(\underset{\substack{\rotatebox{90}{$\in$} \\ z^m}}{\mathbb{C}^\times, \mathbb{C}^\times}) \cong \underset{m}{\mathbb{Z}}$$

$$H_*^{G_\mathcal{O}}(\mathcal{R}) = \bigoplus_{m\in\mathbb{Z}} H_*^{\mathbb{C}^\times}(pt)[z^m] \qquad [z^m]*[z^n] = [z^{m+n}]$$

$$\therefore\ H_*^{G_\mathcal{O}}(\mathcal{R}) \cong \mathbb{C}[\underset{\substack{/\!/ \\ \text{generator of } H^*_{\mathbb{C}^\times}(pt)}}{w}, \underset{\substack{\| \\ [z^{\pm 1}]}}{y^{\pm}}] \qquad \mathrm{Spec} = \mathbb{C}\times\mathbb{C}^\times$$

$\underline{\text{Ex 2}}$ $G=\mathbb{C}^\times$, $N=\mathbb{C}$ natural repr.

$$\mathcal{R}=\coprod_{m\in\mathbb{Z}}\{(z^m,\underset{\in\,\mathbb{C}[[z]]}{f(z)})\mid z^m f(z)\in\mathbb{C}[[z]]\}$$


$\mathcal{R}_m = z^{\max(-m,0)}\mathbb{C}[[z]]$

$$[\mathcal{R}_1]*[\mathcal{R}_{-1}]=\overbrace{\underset{\substack{\uparrow\\ \text{gen. of } H^*_{\mathbb{C}^\times}(pt)}}{w}\cdot[\mathcal{R}_0]}^{\text{unit}}$$

$$\Rightarrow H^{G_O}_*(\mathcal{R})\cong\mathbb{C}[\overset{[\mathcal{R}_{-1}]}{\overset{\shortparallel}{x}},\overset{[\mathcal{R}_1]}{y}]$$

$$\mathrm{Spec}=\mathbb{C}^2$$

∘ quantization

$\mathbb{C}^\times \curvearrowright D$ rotation $G_O\times\mathbb{C}^\times\curvearrowright\mathcal{R}$

$$\mathcal{A}_\hbar := (H^{G_O\times\mathbb{C}^\times}_*(\mathcal{R}),\ \text{conv.}) : \text{quantized Coulomb branch}$$

$$H^*_{\mathbb{C}^\times}(pt)=\mathbb{C}[\hbar]$$

$\therefore$ $\mathbb{C}[\mathcal{M}_C]$ is a Poisson algebra

In fact, $\mathcal{M}_C$: holo symplectic ω on regular locus

∘ integrable system

$$H^*_G(pt)\hookrightarrow H^{G_O}_*(\mathcal{R})$$

$$\therefore\ \mathcal{M}_C\xrightarrow{\varpi}\mathrm{Spec}\,H^*_G(pt)=\mathfrak{g}/\!/G=\mathfrak{t}/W\cong\mathbb{A}^l$$

$$H^*_{G\times\mathbb{C}^*}(pt)\longrightarrow H^{G_O\times\mathbb{C}^*}_*(\mathcal{R})$$

$\uparrow$ commutative subalg.

$\therefore$ $H^*_G(pt)\hookrightarrow H^{G_O}_*(\mathcal{R})$ Poisson commuting

Lemma $t \in \mathbb{A}^l$ generic $\Longrightarrow \varpi^{-1}(t) \cong T^\vee$

(complements of finite union of hyperplanes)

$$(\because) \quad H_*^{T_\mathcal{O}}(\mathcal{R})_t \cong H_*^{T_\mathcal{O}}(\overset{\text{fixed pt}}{\mathcal{R}^t})_t$$

$$\overset{t:\text{generic}}{\|} \quad \mathcal{R}^T = \underset{\underset{\operatorname{Hom}(\mathbb{C}^\times, T)\ :\ \text{coweight lattice}}{\mathrel{\rotatebox{90}{$\simeq$}}}}{\mathrm{Gr}_T} \times N_\mathcal{O}^T$$

$$H_*^{T_\mathcal{O}}(\mathrm{Gr}_T) = H_T^*(pt) \otimes \mathbb{C}[\text{coweight lattice}]$$
$$= \mathbb{C}[\mathfrak{t} \times T^\vee] = \mathbb{C}[T^*T^\vee] \quad //$$

ϖ is an integrable system in Liouville sense.

Cor $\mathcal{M}_C \underset{\text{birat}}{\approx} T^*T^\vee / W$

∘ $H_*^{G_\mathcal{O}}(\mathcal{R})$ is finitely generated

Recall • $\mathcal{R} = \bigcup \mathcal{R}_{\le\lambda}$ and $H_*^{G_\mathcal{O}}(\mathcal{R}) = \bigcup H_*^{G_\mathcal{O}}(\mathcal{R}_{\le\lambda})$

This is a filtered alg.

$$[\mathcal{R}_{\le\lambda}] * [\mathcal{R}_{\le\mu}] = \underset{\underset{\text{explicit}}{\uparrow}}{a_{\lambda\mu}} [\mathcal{R}_{\le\lambda+\mu}] + \text{lower}$$

∴ $\mathcal{M}_C \underset{\text{degenerate}}{\rightsquigarrow}$ explicit algebra